An implicit function theorem for non-smooth maps between Fréchet spaces.

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1 Introduction

In this paper, we prove a "hard" inverse function theorem, that is, an inverse function theorem for maps F which lose derivatives: $F(u)$ is less regular than u. Such theorems have a long history, starting with Kolmogorov in the Soviet Union ([2], [3], [4]) and Nash in the United States ([15]), and it would be impossible, in such a short paper, to give a full account of the developments which have occured since. Important contributions have been made since by Hörmander, Zehnder, Mather, Sergeraert, Tougeron, Hamilton, Hermann, Craig, Dacorogna, Bourgain, Berti and Bolle, and lately by Villani and Mouhot. However, all the results which we are aware of require the function F to be inverted to be at least C^2; in the Kolmogorov-Arnol'd-Moser tradition, for instance, one uses the fast convergence of Newton's method to overcome the loss of derivatives. In contrast, we make no smoothness assumption on F, only that is continuous and Gâteaux-differentiable.

We will overcome the loss of derivatives by using a new version of the "soft" inverse function theorem (between Banach spaces), the proof of which is given in [12], namely:

Theorem 1 *Let X and Y be Banach spaces, with respective norms $\|x\|$ and $\|y\|'$. Let $f: X \to Y$ be continuous and Gâteaux-differentiable, with $f(0) = 0$. Assume that the derivative $Df(x)$ has a right-inverse $[Df(x)]_r^{-1}$, uniformly bounded in a neighbourhood of 0:*

$$Df(x)\,[Df(x)]_r^{-1} h = h$$
$$\sup\left\{\left\|[Df(x)]_r^{-1}\right\| \;\mid\; \|x\| \leq R\right\} < m$$

Then, for every $y \in Y$ with $\|y\|' \leq Rm^{-1}$ there is some $\bar{x} \in X$ with $\|\bar{x}\| \leq R$, such that $f(\bar{x}) = \bar{y}$ and $\|\bar{x}\| \leq m\,\|\bar{y}\|'$.

As we just said, we will make no attempt to review the literature on hard inverse function theorems; see the survey by Hamilton [10] for an account up to

1982. We have drawn inspiration from the version in [1], which itself is inspired from Hörmander's result [11]. We have also learned much from the work on the nonlinear wave equation by Berti and Bolle, [5], [6], [7], [8] whom we thank for extensive discussions.

In the section 2, we state our main result, Theorem 1, and we derive it from an approximation procedure, which is described in Theorem 2. We also given some variants of Theorem 1, for instance an implicit function theorem and we describe a particular case when we can gain some regularity. Theorem 2 is proved in section 3, and the theoretical part is thus complete. The next two sections are devoted to applications. Section 5 revisits the classical isometric imbedding problem, which was the purpose for Nash's original work. This is somewhat academic, since it known now that it can be treated without resorting to a hard inverse function theorem (see [9]), but it gives us the opportunity to show on a simple example how our result improves, for instance, on those of Moser [14].

2 The setting

2.1 The spaces

Let $(X_s,\, \|\cdot\|_s)_{s\geq 0}$ be a scale of Banach spaces:

$$0 \leq s_1 \leq s_2 \Longrightarrow (X_{s_2} \subset X_{s_1} \quad \text{and} \quad \|\cdot\|_{s_1} \leq \|\cdot\|_{s_2})$$

We shall assume that there exists a sequence of projectors $\Pi_N : X_0 \to E_N$ where $E_N \subset \bigcap_{s\geq 0} X_s$ is the range of Π_N, with $\Pi_0 = 0$, $E_N \subset E_{N+1}$ and $\bigcup_{N\geq 1} E_N$ is dense in each space X_s for the norm $\|\cdot\|_s$. We assume that for any finite constant A there is a constant $C_1^A > 0$ such that, for all nonnegative numbers s, d satisfying $s + d \leq A$:

$$\|\Pi_N u\|_{s+d} \leq C_1^A N^d \|u\|_s \tag{1}$$

$$\|(1 - \Pi_N)u\|_s \leq C_1^A N^{-d} \|u\|_{s+d} \tag{2}$$

Note that these properties imply some interpolation inequalities, for $0 \leq t \leq 1$ and $0 \leq s_1\,,\, s_2 \leq A$, and for a new constant $C_A^{(2)}$ (see e.g. [8]):

$$\|x\|_{ts_1+(1-t)s_2} \leq C_2^A \|x\|_{s_1}^t \|x\|_{s_2}^{1-t}\ . \tag{3}$$

If all these properties are satisfied, we shall say that the scale $(X_s)\,,\ s \geq 0$, is *regular*, and we shall refer to the Π_N as *smoothing operators*. Let $(Y_s,\, \|\cdot\|'_s)_{s\geq 0}$ be another regular scale of Banach spaces. We shall denote by $\Pi'_N : Y_0 \to E'_N \subset \bigcap_{s\geq 0} Y_s$ the smoothing operators. In the sequel, $B_s(R)$ (resp. $B'_s(R)$) will denote the open ball of center 0 and radius R in X_s (resp. Y_s)

2.2 The map

Recall that a map $F : X \to Y$, where X and Y are Banach spaces, is *Gâteaux-differentiable* at x if there is a linear map $DF(x) : X \to Y$ such that:

$$\forall h \in X, \quad \lim_{t \to 0} \frac{1}{t} \left[F(x + th) - F(x) \right] = DF(x)\, h$$

In the following, $R > 0$ and $S > 0$ are prescribed, with possibly $S = \infty$

Definition 2 *We shall say that $F : B_0(R) \to Y_0$ is* roughly tame *with loss of regularity μ if:*

(a) *F is continuous and Gâteaux-differentiable from $B_0(R) \cap X_s$ to Y_s for any $s \in [0, S)$.*

(b) *For any $A \in [0, S)$ there is a finite constant K_A such that, for all $s < A$ and $x \in B_0(R)$:*

$$\forall h \in X, \quad \|DF(x)h\|'_s \leq K^A (\|h\|_s + \|x\|_s \|h\|_0) \tag{4}$$

(c) *For $x \in B_0(R) \cap E_N$, the linear maps $L_N(x) : E_N \to E'_N$ defined by $L_N = \Pi'_N DF(x)\,|_{E_N}$ have a right-inverse, denoted by $[L_N(x)]_r^{-1}$. There is a constant $\mu > 0$ and, for any $A \in [0, S)$, a positive constant γ_A, such that, for all $s < A$ and $x \in B_0(R)$ we have:*

$$\forall k \in E'_N, \quad \|[L_N(x)]_r^{-1} k\|_s \leq \frac{1}{\gamma^A} N^\mu (\|k\|'_s + \|x\|_s \|k\|'_0) \tag{5}$$

In our assumptions, there is no regularity loss between x and $F(x)$, or more exactly the regularity loss, if there is one, has been absorbed by translating the indexation of the spaces Y_s. The number S represents the maximum regularity available, and the constant μ may be interpreted as the loss of derivatives incurred when solving the linearized equation $L_N(x)h = k$. Note that we need $\mu < S$ to start the process.

When trying to solve $F(x) = y$, it is thus natural to assume that $y - F(0)$ is small in $Y_\mu$and look for x in X_0. This was done in [12] by assuming that $DF(x)$ has a right inverse which satisfied estimates similar to (4) and (5), but which were independent of the base point x. In the present work, since the tame estimates depend on x with loss of regularity, we will have to assume that y is small in a more regular space Y_δ, with $\delta > \mu$.

2.3 The main result

We will need an assumption relating μ, δ and S with $\mu < \delta$ and $S > \mu$. Here it is:

Condition 3 *There is some κ such that:*

$$1 < \kappa < 2 \quad \text{and} \quad \min\{\kappa^2,\, \kappa+1\}\mu < \delta \tag{6}$$

$$\frac{\kappa^2}{\kappa-1}\mu < S \tag{7}$$

Inequality (7) imposes $S > 4\mu$, since 4 is the minimum value of $\frac{\kappa^2}{\kappa-1}$, attained when $\kappa = 2$. On the other hand, $\min\{\kappa^2,\, \kappa+1\}$ is an increasing function of κ, which coincides with κ^2 when $\kappa \leq \left(1+\sqrt{5}\right)/2$ and coincides with $\kappa+1$ when $\kappa > \left(1+\sqrt{5}\right)/2$. So inequality (6) imposes $\delta > 1$, attained when $\kappa = 1$.

Let us represent condition (3) geometrically. Define a real function φ on $(0,\ 3]$ by:

$$\varphi\left(x\right) = \begin{cases} \frac{1}{2}x\left(1-\sqrt{1-\frac{4}{x}}\right)+1 & \text{if} \quad 4 < x \leq \frac{3+\sqrt{5}}{\sqrt{5}-1} \\ \frac{x^2}{4}\left(1-\sqrt{1-\frac{4}{x}}\right)^2 & \text{if} \qquad x \geq \frac{3+\sqrt{5}}{\sqrt{5}-1} \end{cases} \tag{8}$$

Proposition 4 *$(\mu, \delta, S) \in R_+^2$ satisfies condition (3) if and only if $\frac{\delta}{\mu} \geq 3$ or $\frac{\delta}{\mu} \leq 3$ and $\frac{\delta}{\mu} \geq \varphi\left(\frac{S}{\mu}\right)$*

Proof. Follows immediately from inverting formulas (6) and (7). ■

We have represented the admissible region for $\left(\frac{S}{\mu}, \frac{\delta}{\mu}\right)$ on Figure 1: it is the region Ω above the curve.

Figure 1: The function $\varphi\left(x\right)$

The parameter κ decreases from $\kappa = 2$ (corresponding to $S/\mu = 3$) to $\kappa = 1$ (corresponding to $S/\mu \to \infty$) along the curve. Note the kink at $S/\mu = \frac{3+\sqrt{5}}{\sqrt{5}-1}$, $\delta/\mu = \frac{1}{2}\left(3+\sqrt{5}\right)$ corresponding to $\kappa = \frac{1}{2}\left(1+\sqrt{5}\right)$ (the golden ratio). In the sequel, we will separate the case $\kappa \leq \frac{1+\sqrt{5}}{2}$ (to the right) from the case $\kappa \geq \frac{1+\sqrt{5}}{2}$ (to the left):

$$\begin{array}{ccc} 1 < \kappa \leq \frac{1+\sqrt{5}}{2} & \frac{\delta}{\mu} > \kappa^2 \geq 1 & \frac{S}{\mu} > \frac{\kappa^2}{\kappa-1} \geq \frac{3+\sqrt{5}}{\sqrt{5}-1} \\ \frac{1+\sqrt{5}}{2} \leq \kappa < 2 & \frac{\delta}{\mu} > \kappa+1 \geq \frac{3+\sqrt{5}}{2} & \frac{S}{\mu} > \frac{\kappa^2}{\kappa-1} \geq 4 \end{array}$$

Theorem 5 *Assume $F : B_0(R) \cap X_s \to Y_s$, $0 \leq s < S$, is roughly tame with loss of regularity μ. Suppose $F(0) = 0$. Let $\delta > 0$ and $\alpha > 0$ be such that*

$$\frac{\delta}{\mu} > \varphi\left(\frac{S}{\mu}\right) \tag{9}$$

$$\frac{\alpha}{\mu} < \min\left\{\frac{\delta}{\mu} - \varphi\left(\frac{S}{\mu}\right),\ \frac{S}{\mu} - \varphi^{-1}\left(\frac{\delta}{\mu}\right)\right\} \tag{10}$$

Then one can find $\rho > 0$ and $C > 0$ such that, for any $y \in Y_\delta$ with $\|y\|'_\delta \leq \rho$, there some $x \in X_\alpha$ such that:

$$F(x) = y$$
$$\|x\|_0 \leq 1$$
$$\|x\|_\alpha \leq C\|y\|_\delta$$

It follows that the map F sends X_0 into Y_0, while F^{-1} sends Y_δ into X_0. This parallels the situation with the linearized operator $DF(0)$, which sends X_0 into Y_0 while $DF(0)^{-1}$ sends Y_μ into X_0, with $\mu < \delta$. More precisely, we have:

- if $F(\bar{x}) = \bar{y}$, then $F(X_0)$ contains some δ-neighbourhood of $\bar{y}$
- if $F(\bar{x}) = \bar{y}$, then $F^{-1}(\bar{y} + Y_\delta)$ contains some α-neighbourhood of $\bar{x}$

S is the maximal regularity on x and δ is the minimal regularity on y that we will need in the approximation procedure, bearing in mind that $F(0) = 0$ (see Corollary 6 below for the case when $F(0) = \bar{y} \neq 0$ has fininte regularity). Note that we may have $\delta > S$: this simply means that the right-hand side y is more regular than the sequence of approximate solutions x_n that we will construct.

$\alpha > 0$ is the regularity of the solution x. Note the significance of 9) and (10) take together. The inequality $\frac{\delta}{\mu} > \varphi\left(\frac{S}{\mu}\right)$ tells us that the right-hand side y is more regular than needed (or, alternatively, that the full range of S has not been used), and this "excess regularity", measured by the difference $\frac{\delta}{\mu} - \varphi\left(\frac{S}{\mu}\right)$ (or, alternatively, $\frac{S}{\mu} - \varphi^{-1}\left(\frac{\delta}{\mu}\right)$) can be diverted to x. For instance, if $S/\mu \to \infty$, the total loss of regularity $\delta - \alpha$ between y and x satisfies

$$\delta - \alpha > \mu\varphi\left(\frac{S}{\mu}\right)$$

and can be made as close to μ as one wishes: we can start the iterative procedure x_n from a very regular initial point. However, we lose control on ρ (which goes to zero) and C (which goes to infinity). On the other hand, when $\delta/\mu > 3$, that is, when we are not worried about the loss of regularity, then S can be any number larger than μ, that is, we need very little regularity to start with.

We now go from the case $F(0) = 0$ to the case $F(\bar{x}) = \bar{y}$. There is some subtelty there because 0 belong to all the X_s, while $\bar{y}$ does not, and puts adittional limits to the regularity. We shall say that F is roughly tame at $\bar{x}$ if $F(x - \bar{x})$ is roughly tame at 0.

Corollary 6 *Suppose $\bar{x} \in X_{S_1}$, $\bar{y} \in Y_{S_2}$ and $F(\bar{x}) = \bar{y}$. Assume $F(x)$ sends $X_s \cap B_0(\bar{x}, R)$ into Y_s for every $s \geq 0$, and is roughly tame at $\bar{x}$ with loss of regularity μ. Set $S = \min\{S_1, S_2\}$. Let δ and α satisfy (9) and (10). Then one can find $\rho > 0$ and $C > 0$ such that, for any y with $\|\bar{y} - y\|'_\delta \leq \rho$, there is a solution x of the equation $F(x) = y$, with $\|x - \bar{x}\|_0 \leq 1$ and $\|x - \bar{x}\|_\alpha \leq C\|y - \bar{y}\|'_\delta$.*

Proof. Consider the map $\Phi(x) := F(x + \bar{x}) - \bar{y}$. It is roughly tame, with $F(0) = 0$, and we can apply the preceding Theorem with $S = \min\{S_1, S_2\}$. The result follows ■

We now deduce an implicit function theorem. Let V be a Banach space and let $F : B(R, X_0 \times V) \cap (X_s \times V) \to Y_s$, $0 \leq s < S$ satisfy the following:

Definition 7 **(a')** *F is continuous and Gâteaux-differentiable for any $s \in [0, S)$. We write:*

$$DF(x, v) = (D_x F(x, v),\ D_v F(x, v))$$

(b') *For any $A \in [0, S)$ there is a finite constant K_A such that, for all $s < A$ and $(x, v) \in B(R, X_0 \times V)$:*

$$\forall h \in X, \quad \|DF(x, v)h\|'_s \leq K^A(\|h\|_s + \|x\|_s \|h\|_0)$$

(c') *For $x \in (x, v) \in B(R, X_0 \times V) \cap (E_N \times V)$, the linear maps $L_N(x, v)$: $E_N \to E'_N$ defined by $L_N = \Pi'_N D_x F(x, v)|_{E_N}$ have a right-inverse, denoted by $[L_N(x, v)]_r^{-1}$. There is a constant $\mu > 0$ and, for any $A \in [0, S)$, a positive constant γ_A, such that, for all $s < A$ and $(x, v) \in B(R, X_0 \times V)$ we have:*

$$\forall k \in E'_N, \quad \|[L_N(x, v)]_r^{-1} k\|_s \leq \frac{1}{\gamma^A} N^\mu (\|k\|'_s + \|x\|_s \|k\|'_0)$$

Corollary 8 *Assume (a'), (b'), (c') are satisfied and $F(0, 0) = 0$. Take any α with $0 < \alpha < S - 4\mu$. Then one can find $\rho > 0$ and $C > 0$ such that, for any v with $\|v\| \leq \rho$, there is a some x such that:*

$$F(x, v) = 0$$
$$\|x\|_0 \leq 1$$
$$\|x\|_\alpha \leq C\|v\|$$

Proof. Consider the Banach scale $X_s \times V$ and $Y_s \times V$ with the natural norms. Consider the map $\Phi(x, v) = (F(x, v), \ v)$ from $X_s \times V$, $0 \leq s < S$, into $Y_s \times V$. It is roughly tame with $\Phi(0,0) = (0,0)$ and we can apply the preceding Theorem with $\delta = \infty$. Condition (10) becomes

$$\frac{\alpha}{\mu} < \frac{S}{\mu} - 4$$

■

2.4 A particular case

In the case when $F(x) = Ax + G(x)$ where A is linear, we can improve the regularity.

Proposition 9 *Suppose $F(x) = Ax + G(x)$, where $A : X_{s+\nu} \to Y_s$ is a continuous linear operator, independent of the base point x, and G satisfies $G(0) = 0$. Suppose moreover that:*

(a) *G is continuous and Gâteaux-differentiable from $B_0(R) \cap X_s$ to Y_s for any $s \in [0, \ S)$.*

(b) *For any $A \in [0, \ S)$ there is a finite constant K_A such that, for all $s \leq A$ and $x \in B_0(R)$:*

$$\forall h \in X, \quad \|DG(x)h\|'_s \leq K^A(\|h\|_s + \|x\|_s\|h\|_0)$$

(c) *For $x \in B_0(R) \cap E_N$, the linear maps $L_N(x) : E_N \to E'_N$ have a right-inverse, denoted by $[L_N(x)]_r^{-1}$. There is a constant $\mu > 0$ and, for any $A \in [0, \ S)$, a positive constant γ_A, such that, for all $s \in [0, \ S)$ and $x \in B_0(R)$ we have:*

$$\forall k \in E'_N, \quad \|[L_N(x)]_r^{-1}k\|_s \leq \frac{1}{\gamma^A} N^\mu(\|k\|'_s + \|x\|_s\|k\|'_0)$$

(d) *The E_N are A-invariant.*

Let $\delta > 0$ and $\alpha > 0$ be such that

$$\frac{\delta}{\mu} > \varphi\left(\frac{S}{\mu}\right)$$
$$\frac{\alpha}{\mu} < \min\left\{\frac{\delta}{\mu} - \varphi\left(\frac{S}{\mu}\right), \ \frac{S}{\mu} - \varphi^{-1}\left(\frac{\delta}{\mu}\right)\right\}$$

. Then one can find $\rho > 0$ and $C > 0$ such that, for any $y \in Y_\delta$ with $\|y\|'_\delta \leq \rho$, there some $x \in X_\alpha$ such that:

$$F(x) = y$$
$$\|x\|_0 \leq 1$$
$$\|x\|_\alpha \leq C\|y\|_\delta$$

In this situation, a direct application of Theorem 5 would give a loss or regularity of $\mu+\nu$. Proposition 9 tells us that the μ is enough: the loss of regularity due to the linear part can be circumventend.

2.5 The approximating sequence

Theorem 5 is proved by an approximation procedure: we construct by induction a sequence x_n having certain properties, and we show that it converges to the desired solution. We now describe that sequence, and give the proof of convergence. The actual construction of the sequence is postponed to the next section.

Given an integer N and a real number $\alpha>1$, we shall denote by $E[N^\alpha]$ the integer part of N^α:

$$E[N^\alpha]\leq N^\alpha<E[N^\alpha]+1$$

Theorem 10 *Assume that μ,δ,S and κ satisfy condition* **??**. *Choose σ and β such that:*

$$\frac{\kappa^2}{\kappa-1}\mu<\kappa\beta<\sigma<S \tag{11}$$

Impose, moreover:

- *For* $1<\kappa\leq\frac{1+\sqrt{5}}{2}$
$$\kappa\beta>\kappa\mu+\sigma-\frac{\delta}{\kappa} \tag{12}$$
- *For* $\frac{1+\sqrt{5}}{2}\leq\kappa<2$
$$\beta>\mu+\sigma-\delta \tag{13}$$

Then one can find $N_0\geq 2$, $\rho>0$ and $c>0$ such that, for any $y\in Y$ with $\|y\|'_\delta\leq\rho$, there are sequences $(x_n)_{n\geq1}$ in $B_0(1)$ and $N_n:=N_0^{(\kappa n)}$ satisfying:

- *For* $1<\kappa\leq\frac{1+\sqrt{5}}{2}$,
$$\Pi'_{N_n}F(x_n)=\Pi'_{N_{n-1}}y \text{ and } x_n\in E_{N_n} \tag{14}$$
- *For* $\frac{1+\sqrt{5}}{2}\leq\kappa<2$
$$\Pi'_{N_n}F(x_n)=\Pi'_{N_n}y \text{ and } x_n\in E_{N_n} \tag{15}$$

And in both cases:

$$\|x_1\|_0\leq cN_1^{\mu}\|y\|'_\delta \text{ and } \|x_{n+1}-x_n\|_0\leq c\,N_n^{\kappa\beta-\sigma}\|y\|'_\delta \tag{16}$$
$$\|x_1\|_\sigma\leq cN_1^{\beta}\|y\|'_\delta \text{ and } \|x_{n+1}-x_n\|_\sigma\leq c\,N_n^{\kappa\beta}\|y\|'_\delta \tag{17}$$

The set of admissible σ and β is non-empty. Indeed, because of (7), we have $\frac{\kappa^2}{\kappa-1}\mu < S$, so we can find $\kappa\beta$ and σ satisfying condition (11). For $1 < \kappa \leq \left(1+\sqrt{5}\right)/2$, we have $\delta > \mu\kappa^2$, so $\kappa\mu - \delta/\kappa < 0$ and $\kappa\beta$ can satisfy both (11) and (12). For $\kappa \geq \left(1+\sqrt{5}\right)/2$, we have $\delta - \mu > \kappa\mu$, so $\mu + \sigma - \delta < \sigma - \kappa\mu$ and condition (13) is satisfied provided $\beta > \sigma - \kappa\mu$, or $\kappa\beta > \kappa\sigma - \kappa^2\mu$. If σ satisfies (11), we have $\kappa\sigma - \kappa^2\mu > (\kappa - 1)\,\sigma > 0$, so we can find $\kappa\beta$ satisfying (11) and (13).

Note that the estimate on $\|x_{n+1} - x_n\|_\sigma$ blows up very fast when $n \to \infty$, while the estimate on $\|x_{n+1} - x_n\|_0$ goes to zero very fast, since $\kappa\beta - \sigma < 0$. Using the interpolation inequality (3), this will enable us to maintain control of some intermediate norms.

The proof of Theorem 10 is postponed to the next section. We now show that it implies Theorem 5. Let u begin with an estimate:

Lemma 11 *Given $0 < A < S$ there is a constant C_3^A such that, for all $s \in [0, A]$, all integers N, $P \geq 0$ and any $x \in B_0(1) \cap X_s$:*

$$\begin{aligned}
\|F(x)\|'_s &\leq C_3^A \|x\|_s \\
\|(1 - \Pi'_N)F(x)\|'_0 &\leq C_3^A N^{-s} \|x\|_s \\
\|\Pi'_{N+P}(1 - \Pi'_N)F(x)\|'_0 &\leq C_3^A N^{-s} \|x\|_s
\end{aligned}$$

Proof. The function $\varphi :\, t \in [0, 1] \to F(tx)$ has derivative $\frac{d}{dt}\varphi = DF(tx)x$ and by the tame estimates (4) on $DF(x)$, we have $\|\frac{d}{dt}\varphi(t)\|'_s \leq 2K_A\|x\|_s$. Since $\varphi(0) = 0$, this gives our first estimate. Combining it with (2), we get the second one, and applying (1) with $d = 0$, we get the third one. ∎

Let us now prove Theorem 5. Since $\kappa\beta - \sigma < 0$, the inequalities (16) imply that the sequence (x_n) is Cauchy in X_0, and has a limit $\bar{x}$ with $\|\bar{x}\|_0 \leq C\|\bar{y}\|'_\delta$, where

$$C = c(N_1^\mu + \sum_{n\geq 1} N_n^{\kappa\beta - \sigma})$$

Then $F(x_n)$ converges to $F(\bar{x})$ in Y_0 , by the continuity of $F : X_s \to Y_s$.Similarly, (17) implies that $\|x_n\|_\sigma \leq C'_n N_n^{\kappa\beta}\|\bar{y}\|'_\delta$, with:

$$C'_n := cN_n^{-\kappa\beta}\left(N_1^\beta + \sum_{i=1}^{n-1} N_i^{\kappa\beta}\right)$$

and $C' := \sup_n C'_n < \infty$, so that $\|x_n\|_\sigma \leq C' N_n^{\kappa\beta}\|\bar{y}\|'_\delta$ for all n.

The case $1 < \kappa \leq \frac{1+\sqrt{5}}{2}$. We have, by (14):

$$F(x_n) = (1 - \Pi'_{N_n})F(x_n) + \Pi'_{N_{n-1}}\bar{y}\,. \tag{18}$$

Then Lemma 11 gives the estimate

$$\|(1 - \Pi'_{N_n})F(x_n)\|'_0 \leq C_\delta^{(3)} N_n^{-s}\|x_n\|_s \quad \text{for all} \;\; s < \delta$$

Substituting $\|x_n\|_\sigma \leq C' N_n^{\kappa\beta} \|y\|'_\delta$, we get:

$$\|(1 - \Pi'_{N_n}) F(x_n)\|'_0 \leq C' \, C_\delta^{(3)} \, N_n^{\kappa\beta - \sigma} \|y\|'_\delta \quad \text{for all} \quad s \leq \delta$$

By (11), the exponent $(\kappa\beta - \sigma)$ is negative. So $(1 - \Pi'_{N_n}) F(x_n)$ converges to zero in Y_0. Now, using the inequality (2) we get

$$\|(1 - \Pi'_{N_{n-1}}) \bar{y}\|'_0 \leq C_\delta^{(1)} N_{n-1}^{-\delta} \|\bar{y}\|'_\delta \quad \text{for all} \quad s \leq \delta$$

so $\Pi'_{N_{n-1}} \bar{y}$ converges to $\bar{y}$ in Y_0. So both terms on the right-hand side of (18) converge to zero, and we get $F(\bar{x}) = \bar{y}$, as announced.

Together with the interpolation inequality (3), conditions (16) and (17) imply:

$$\|x_{n+1} - x_n\|_{(1-t)\sigma} \leq c_t N_n^{\kappa\beta - t\sigma} \, \|y\|'_\delta$$

The exponent on the right-hand side is negative for $t > \kappa\beta/\sigma$, so that $(1-t)\,\sigma < \sigma - \kappa\beta$. Arguing as above, if follows that $\|\bar{x}\|_\alpha \leq C \, \|y\|'_\delta$, provided:

$$\alpha < \sup_{\mathcal{A}_1} \{\sigma - \kappa\beta \ \}$$

where:

$$\mathcal{A}_1 = \left\{ (\kappa, \beta, \sigma) \mid \quad \begin{array}{c} \sigma - \kappa\beta < \frac{\delta}{\kappa} - \kappa\mu \\ \frac{\kappa^2}{\kappa - 1}\mu < \kappa\beta < \sigma < S \end{array} \ \right\}$$

Set $\alpha' = \alpha/\mu$, $\sigma' = \sigma/\mu$, $\beta' = \beta/\mu$, $\delta. = \delta/\mu$, $S' = S/\mu$. The problem becomes:

$$\alpha' < \sup_{\mathcal{A}'_1} \{\sigma' - \kappa\beta' \ \}$$

$$\mathcal{A}'_1 = \left\{ (\kappa, \beta', \sigma') \mid \quad \begin{array}{c} \sigma' - \kappa\beta' < \frac{\delta'}{\kappa} - \kappa \\ \frac{\kappa^2}{\kappa - 1} < \kappa\beta' < \sigma' < S' \end{array} \ \right\}$$

For given κ, Figure 3 gives the admissible (β, σ) region in the case $S' > \delta'/\kappa - \kappa$ (upper horizontal line) and in the case $S' < \delta'/\kappa - \kappa$ (lower horizontal line). The admissible region is to the right of the vertical $\beta' = \kappa/\,(\kappa - 1)$, both in the case $S' > \delta'/\kappa - \kappa$ (right line) and in the case $S' < \delta'/\kappa - \kappa$ (left line)

Figure 3: The admissible (β, σ) region in the first case

The maximum is attained at the upper left corner of the admissible region, which is the point $(\beta', \min\{S', \ \kappa\beta' - \kappa + \delta'/\kappa\})$, with $\beta' = \kappa\,(\kappa-1)^{-1}$. Hence:

$$\sup_{\mathcal{A}_1}\{\sigma' - \kappa\beta'\} = \min\left\{S' - \frac{\kappa^2}{\kappa-1}, \ \frac{\delta'}{\kappa} - \kappa\right\} \tag{19}$$

The case $\frac{1+\sqrt{5}}{2} \leq \kappa < 2$ The argument is the same, except that we have to replace $\Pi'_{N_{n-1}}\bar{y}$ by $\Pi'_{N_n}\bar{y}$ in (18).

We now have:

$$\alpha' < \sup_{\mathcal{A}'_2}\{\sigma' - \kappa\beta' \ \}$$

$$\mathcal{A}'_2 = \left\{(\kappa, \beta', \sigma') \mid \begin{array}{c} \sigma' < \beta' + \delta - 1 \\ \frac{\kappa^2}{\kappa-1} < \kappa\beta' < \sigma' < S' \end{array}\right\}$$

For given κ, the admissible (β, σ) region is given in Figure 4, in the case $S > \delta - 1 + \kappa/\left(\kappa-1\right)$ (upper horizontal line) and in the case $S < \delta - 1 + \kappa/\left(\kappa-1\right)$ (lower horizontal line). The vertical is $\beta = \kappa/\left(\kappa-1\right)$.

Figure 4: The admissible (β, σ) region in the second case

Again the maximum is attained in the upper left corner, which is the point $(\beta',\ \min\{S',\ \delta' - 1 + \beta'\})$ with $\beta' = \kappa\,(\kappa - 1)$. This gives:

$$\sup_{\mathcal{A}_2} \{\sigma' - \kappa\beta'\ \} = \min\left\{S' - \frac{\kappa^2}{\kappa - 1},\ \delta' - 1 - \kappa\right\} \tag{20}$$

Putting (19) and (20) together gives formula (10)

3 Proof of Theorem 2

We work under the assumptions of Theorem 10. So $\mu, \delta, S, \kappa, \sigma, \beta, \bar{y}$ are given. Note that we may have $\sigma < \delta$.

We assume $\bar{y} \neq 0$ (the case $\bar{y} = 0$ is obvious). We fix $A = \sigma$, and the constants $C_1^A,\ C_2^A,\ C_3^A,\ K^A,\ \gamma^A$ of (1, 2, 3, 4, 5) and Lemma 11 are simply denoted $C_1,\ C_2^A,\ C_3^A,\ K,\ \gamma$. The proof will make use of a certain number of constants, which we list here to make sure that they do not depend on the iteration step and can be fixed at the beginning.

Recall that $2^{(\alpha)}$ is the integer part of 2^α, and set $P_n = E\left[2^{\kappa^n}\right]$. There is a constant $g > 1$ such that for all $N_0 \geq 2$ and $n \geq 0$,

$$g^{-1} P_n^\kappa \leq P_{n+1} \leq g\, P_n^\kappa \tag{21}$$

We define constants $B_0,\ B_1$ and B_2 by:

$$B_0 = (P_1 + \sum_{n\geq 1} P_n^{\kappa\beta-\sigma})^{-1} \tag{22}$$

$$B_1 := \sup_n \{P_n^{-\beta}(P_1^{\beta} + \sum_{1\leq i\leq n-1} P_i^{\kappa\beta}) \mid n \geq 1\} \tag{23}$$

$$B^2 := \sup_n \left\{ B_1 P_n^{-(\kappa-1)\beta} + 1 \mid n \geq 1 \right\} \tag{24}$$

We shall use Theorem 1 to construct inductively the sequence x_n, thanks to a sequence of carefully chosen norms. For this purpose, we will have to take c large and ρ small.

4 Choice of N_0

For $1 < \kappa \leq \frac{1+\sqrt{5}}{2}$, we consider the following function of n:

$$\varphi_2(n) := 2\, B_1 C_3\, c\, P_n^{\beta-\kappa(\beta-\mu)} + C_1 \left(P_n^{\sigma-\delta/\kappa-\kappa(\beta-\mu)} g^{\delta/\kappa} + P_n^{(\sigma-\delta)_+ - (\delta-\sigma)_+/\kappa - \kappa(\beta-\mu)} \right) \tag{25}$$

By condition (11) and (12), all the exponents are negative. So we may pick n_0 so large that:

$$\varphi_2(n_0) \leq \gamma c\, (B_2+2)^{-1}\, g^{-\mu} \text{ for all } n \geq n_0 \tag{26}$$

For $\frac{1+\sqrt{5}}{2} \leq \kappa < 2$, we consider the following function of N_0 and n:

$$\varphi_1(n_0) := 2\, B_1 C_3\, c\, P_n^{\beta-\kappa(\beta-\mu)} + C_1 \left(P_n^{\sigma-\delta-\kappa(\beta-\mu)} + g^{(\sigma-\delta)_+} P_n^{\kappa(\sigma-\delta)_+ - (\delta-\sigma)_+ - \kappa(\beta-\mu)} \right) \tag{27}$$

By condition (11) and (13), all the exponents are negative. So we may pick N_0 so large that:

$$\varphi_1(n_0) \leq \gamma c\, (B_2+2)^{-1}\, g^{-\mu} \text{ for all } n \geq n_0 \tag{28}$$

In both cases we set $N_0 = P_{n_0}$, and $N_n = E\left[N_0^{n^\kappa}\right] = E\left[2^{(n_o n)^\kappa}\right]$. So the expressions (25) and (26) are less than $\gamma c\,(B_2+2)^{-1} g^{-\mu}$ when one substitutes N_n for P_n.

4.1 Construction of the initial point

The case $1 < \kappa \leq \frac{1+\sqrt{5}}{2}$. Thanks to inequality (2), $\|\Pi'_{N_0}\bar{y}\|_0 \leq C_1 \|\bar{y}\|'_\delta$ and $\|\Pi'_{N_0}\bar{y}\|_\sigma \leq C_1 N_0^{(\sigma-\delta)_+} \|\bar{y}\|'_\delta$, where t_+ denotes the positive part of the real number t. We choose the norm

$$\mathcal{N}_0(x) = \|x\|_0 + N_0^{-(\sigma-\delta)_+} \|x\|_\sigma$$

on E_{N_1} and the norm:

$$\mathcal{N}_0'(y) = \|y\|_0' + N_0^{-(\sigma-\delta)_+} \|y\|_\sigma'$$

on E_{N_1}'. For these norms, E_{N_1} and E_{N_1}' are Banach spaces. Note that

$$\mathcal{N}_0'(\Pi_{N_0}' y) < 2C_1 \|y\|_\delta \quad \text{for } y \in E_{N_1}' \tag{29}$$

For $\mathcal{N}_0(x) \leq 1$, we define

$$f(x) := \Pi_{N_1}' F(x) \in E_{N_1}'$$

The function f is continuous and Gâteaux-differentiable for the norms $\mathcal{N}_0$ and $\mathcal{N}_0'$, with $f(0) = 0$. Moreover, using the tame estimate (5) and applying assumption (1) to $\|x\|_\sigma$, we find that:

$$\sup\left\{\left\| [Df(x)]^{-1} k\right\|_0 \mid \mathcal{N}(x) \leq 1\right\} < \frac{2N_1^\mu}{\gamma} \|k\|_0' \tag{30}$$

$$\sup\left\{\left\| [Df(x)]^{-1} k\right\|_\sigma \mid \mathcal{N}(x) \leq 1\right\} < \frac{N_1^\mu}{\gamma} (\|k\|_\sigma' + N_0^{(\sigma-\delta)_+} \|k\|_0') \tag{31}$$

hence:

$$\sup\left\{\mathcal{N}_0([Df(x)]^{-1} k) \mid \mathcal{N}(x) \leq 1\right\} < \frac{3N_1^\mu}{\gamma} \mathcal{N}_0'(k) \tag{32}$$

By Theorem 1, we can solve $f(\bar{u}) = \bar{v}$ with $\mathcal{N}_0(\bar{u}) \leq 1$ if $\mathcal{N}_0'(\Pi_{N_0}' \bar{y}) \leq \gamma\left(3N_1^\mu\right)^{-1}$. By (29), this is fulfilled provided:

$$\|\bar{y}\|_\delta' \leq \frac{\gamma}{6C_1 N_1^\mu} =: \rho\,. \tag{33}$$

In addition, Theorem 1 tells us that we have the estimate:

$$\mathcal{N}_0(\bar{u}) \leq 3N_1^\mu \gamma^{-1} \mathcal{N}_0'\left(\Pi_{N_0}' \bar{y}\right) \leq 6C^{(1)} N_1^\mu \gamma^{-1} \|\bar{y}\|_\delta' \tag{34}$$

If (33 is satisfied, $x_1 := \bar{u}$ is the desired solution in E_{N_1} of the projected equation $\Pi_{N_1}' F(x_1) = \Pi_{N_0}' \bar{y}$, with $\mathcal{N}_0(\bar{u}) \leq 1$. Let us check conditions (17) and (16). We have, by (34):

$$\mathcal{N}_0(x_1) = \|x_1\|_0 + N_0^{-(\sigma-\delta)_+} \|x_1\|_\sigma \leq R = 6C^{(1)} N_1^\mu \gamma^{-1} \|\bar{y}\|_\delta'$$

Since $N_0 \leq g^{1/\kappa} N_1^{1/\kappa}$, we find:

$$\|x_1\|_0 + (gN_1)^{-\kappa^{-1}(\sigma-\delta)_+} \|x_1\|_\sigma \leq 6C^{(1)} N_1^\mu \gamma^{-1} \|\bar{y}\|_\delta'$$

Since $\mu + \kappa^{-1}(\sigma - \delta)_+ < \beta$, this yields $\|x_1\|_0 \leq cN_1^\mu \|\bar{y}\|_\delta'$ and $\|x_1\|_\sigma \leq cN_1^\beta \|\bar{y}\|_\delta'$ as required, with

$$c := 6C^{(1)} g^{(\sigma-\delta)_+/\kappa} \gamma^{-1} \tag{35}$$

The case $\frac{1+\sqrt{5}}{2} \leq \kappa < 2$ Very few modifications are needed in the above arguments. Replace N_0 by N_1, so that the norms become:

$$\mathcal{N}_0(x) = \|x\|_0 + N_1^{-(\sigma-\delta)_+} \|x\|_\sigma$$
$$\mathcal{N}_0'(y) = \|y\|_0' + N_1^{-(\sigma-\delta)_+} \|y\|_\sigma'$$

and define as above $f(x) := \Pi_{N_1}' F(x) \in E_{N_1}'$. Because (14) is replaced by (15), we now consider $\Pi_{N_1}' \bar{y} \in E_{N_1}'$. Estimates (29) and (32) still hold. Using Theorem 1 as before, we will be able do find some $\bar{u} \in E_{N_1}$ with $\mathcal{N}_0(\bar{u}) \leq 1$ solving $f(\bar{u}) = \Pi_{N_1}' \bar{y}$ provided $\bar{y}$ satisfies (33). The estimate (34) still holds:

$$\mathcal{N}_0(x_1) = \|x_1\|_0 + N_1^{-(\sigma-\delta)_+} \|x_1\|_\sigma \leq 6C_1 N_1^{\mu} \gamma^{-1} \|\bar{y}\|_\delta'$$

Since $\mu + (\sigma - \delta)_+ < \beta$, this yields $\|x_1\|_0 \leq cN_1^{\mu} \|\bar{y}\|_\delta'$ and $\|x_1\|_\sigma \leq cN_1^{\beta} \|\bar{y}\|_\delta'$ as above, with

$$c := 6C^{(1)} \gamma^{-1} \tag{36}$$

Diminishing ρ, if necessary, we can always assume that, in both cases, ρ and c satisfy the constraint, where B_0 is defined by (22):

$$c\rho < B^{(0)} \tag{37}$$

4.2 Induction

The case $1 < \kappa \leq \frac{1+\sqrt{5}}{2}$. Assume that the result has been proved up to n. In other words, define c by (35), and assume we have found ρ with $c\rho < B^{(0)}$ such that, for $\bar{y} \in Y$ with $\|\bar{y}\|_\delta' \leq \rho$, there exists a sequence $x_1, \cdots, x_n$ satisfying (14), (17), and (16). To be precise:

$$\|x_{i+1} - x_i\|_0 \leq c\, N_i^{\kappa\beta-\sigma} \|\bar{y}\|_\delta' \quad \text{for } i \leq n-1 \tag{38}$$
$$\|x_{i+1} - x_i\|_\sigma \leq c\, N_i^{\kappa\beta} \|\bar{y}\|_\delta' \quad \text{for } i \leq n-1 \tag{39}$$

Since $x_1, \cdots, x_n$ satisfy (16), and $\|\bar{y}\|_\delta' \leq \rho$, this will imply that $\|x_n\|_0 \leq 1 - \eta_n$ with:

$$\eta_n := \frac{\sum_{i \geq n} N_i^{\kappa\beta-\sigma}}{N_1^{\mu} + \sum_{n \geq 1} N_n^{\kappa\beta-\sigma}} \leq c\rho \sum_{i \geq n} N_i^{\kappa\beta-\sigma}\,. \tag{40}$$

Since $x_1, \cdots, x_n$ satisfy (17), we also have:

$$\|x_n\|_\sigma \leq c\,(N_1^{\beta} + \sum_{1 \leq i \leq n-1} N_i^{\kappa\beta}) \|\bar{y}\|_\delta'$$

Using the constant B_1 defined in (22), this becomes:

$$\|x_n\|_\sigma \leq B_1\, c\, N_n^{\beta} \|\bar{y}\|_\delta \tag{41}$$

We are going to construct x_{n+1} so that (14), (17), and (16) hold for $i \leq n$. Write:

$$x_{n+1} = x_n + \Delta x_n$$

By the induction hypothesis, $x_n \in E_{N_n}$ and $\Pi'_{N_n} F(x_n) = \Pi'_{N_{n-1}} \bar{y}$. The equation to be solved by Δx_n may be written in the following form:

$$f_n\left(\Delta x_n\right) = e_n + \Delta y_{n-1} \tag{42}$$
$$f_n(u) := \Pi_{N_{n+1}}\left(F(x_n + u) - F(x_n)\right) \tag{43}$$
$$e_n := \Pi_{N_{n+1}}(\Pi_{N_n} - 1)F(x_n) \tag{44}$$
$$\Delta y_{n-1} := \Pi_{N_n}(1 - \Pi_{N_{n-1}})\bar{y} \tag{45}$$

The function f_n is continuous and Gâteaux-differentiable with $f\left(0\right) = 0$. We will solve equation (42) by applying Theorem 1. We choose the norms:

$$\mathcal{N}_n(u) = \|u\|_0 + N_n^{-\sigma}\|u\|_\sigma \quad \text{on} \quad E_{N_{n+1}} \tag{46}$$
$$\mathcal{N}'_n(v) = \|v\|'_0 + N_n^{-\sigma}\|v\|'_\sigma \quad \text{on} \quad E'_{N_{n+1}} \tag{47}$$

Let $R_n := cN_n^{\kappa\beta-\sigma}\|\bar{y}\|_\delta$. If $\mathcal{N}_n(u) \leq R_n$, we have, by (40)

$$\|u\|_0 \leq \mathcal{N}_n(u) < R_n < cN_n^{\kappa\beta-\sigma}\rho < \eta_n$$

so that $\left\|x_n + u\right\|_0 \leq 1$ and the function f_n is well-defined by (43). Using (41) and (46), we find that, for $\mathcal{N}_n(u) \leq R_n$, we have $\left\|u\right\|_\sigma \leq N_n^\sigma R_n = cN_n^{\kappa\beta}\|\bar{y}\|_\delta$, and hence, with the constants B_1 and B_2defined by (23) and (24):

$$\|x_n + u\|_\sigma \leq c\,(B_1\,N_n^\beta + N_n^{\kappa\beta})\|\bar{y}\|_\delta \leq B_2\,c\,N_n^{\kappa\beta}\|\bar{y}\|'_\delta \tag{48}$$

Plugging this into the tame inequality (5), and taking into account that $c\|\bar{y}\|_\delta = R_n N_n^{\sigma-\kappa\beta}$ then gives:

$$\sup\left\{\left\|\,[Df_n(u)]^{-1}k\right\|_0 \;\middle|\; \mathcal{N}_n(u) \leq R_n\right\} < 2N_{n+1}^\mu\gamma^{-1}\|k\|'_0 \tag{49}$$
$$\begin{aligned}\sup\left\{\left\|\,[Df_n(u)]^{-1}k\right\|_\sigma \;\middle|\; \mathcal{N}_n(u) \leq R_n\right\} &< N_{n+1}^\mu\gamma^{-1}(\|k\|'_\sigma + B_2cN_n^{\kappa\beta}\|k\|'_0\|\bar{y}\|'_\delta) \\ &= N_{n+1}^\mu\gamma^{-1}(\|k\|'_\sigma + B_2\,R_n\,N_n^\sigma\|k\|'_0)\end{aligned} \tag{50}$$

Hence:

$$\sup\left\{\mathcal{N}_n([Df_n(u)]^{-1}k) \mid \mathcal{N}_n(u) \leq R_n\right\} < \gamma^{-1}\left(B^{(2)} + 2\right)N_{n+1}^\mu\mathcal{N}'_n(k)$$

By Theorem 1, we will be able to solve (42) with $\mathcal{N}_n(\Delta x_n) \leq R_n$ provided:

$$\mathcal{N}'_n\left(e_n + \Delta y_{n-1}\right) \leq \gamma\left(B^{(2)} + 2\right)^{-1} N_{n+1}^{-\mu}R_n = \gamma c\left(B^{(2)} + 2\right)^{-1} N_{n+1}^{-\mu}N_n^{\kappa\beta-\sigma}\|\bar{y}\|_\delta \tag{51}$$

We now compute $\mathcal{N}_n'(e_n + \Delta y_{n-1})$. Applying inequality (2) to (45), we get:

$$\|\Delta y_{n-1}\|_0' \leq C_1 N_{n-1}^{-\delta} \|\bar{y}\|_\delta' \tag{52}$$

$$\|\Delta y_{n-1}\|_\sigma' \leq C_1 N_{n-1}^{-(\delta-\sigma)_+} N_n^{(\sigma-\delta)_+} \|\bar{y}\|_\delta' \tag{53}$$

Because of the induction hypothesis, we get

$$\|x_n\|_\sigma \leq c\,(N_1^\beta + \sum_{1\leq i\leq n-1} N_i^{\kappa\beta}) \|\bar{y}\|_\delta' \leq B_1\, c\, N_n^\beta \|\bar{y}\|_\delta' \tag{54}$$

So, combining Lemma 11 with conditions (17) and (16), we find that

$$\|e_n\|_0' \leq C^{(3)} N^{-\sigma} \|x_n\|_\sigma \leq B_1 C_3\, c\, N_n^{\beta-\sigma} \|\bar{y}\|_\delta' \tag{55}$$

$$\|e_n\|_\sigma' \leq C^{(3)} \|x_n\|_\sigma \leq B_1 C_3\, c\, N_n^{\beta} \|\bar{y}\|_\delta' \tag{56}$$

We now check (51). Using the estimates (52), (53), (55), (56), and remembering (21), we have :

$$\mathcal{N}_n'(e_n) \leq 2 B_1 C_3\, c\, N_n^{\beta-\sigma} \|\bar{y}\|_\delta' \tag{57}$$

$$\mathcal{N}_n'(\Delta y_{n-1}) \leq C_1 \|\bar{y}\|_\delta' \left(N_{n-1}^{-\delta} + N_{n-1}^{-\sigma} N_{n-1}^{-(\delta-\sigma)_+} N_n^{(\sigma-\delta)_+} \right) \tag{58}$$

$$\mathcal{N}_n'(e_n) + \mathcal{N}_n'(\Delta y_{n-1}) \leq \left[2\, B_1 C_3\, c\, N_n^{\beta-\sigma} + C_1 \left(N_{n-1}^{-\delta} + N_{n-1}^{-\sigma-(\delta-\sigma)_+} N_n^{(\sigma-\delta)_+} \right) \right] \|\bar{y}\|_\delta' \tag{59}$$

Condition (51) is satisfied provided:

$$2\, B_1 C_3\, c\, N_n^\beta + C_1 \left(N_n^{\sigma-\delta/\kappa} g^{\delta/\kappa} + N_n^{(\sigma-\delta)_+ - (\delta-\sigma)_+/\kappa} \right) \leq \gamma c\, (B_2+2)^{-1}\, g^{-\mu} N_n^{\kappa(\beta-\mu)} \tag{60}$$

Dividing by $N_n^{\kappa(\beta-\mu)}$ on both sides, we find that the sequence on the left-hand side is just a subsequence of $\varphi_2(n)$, where φ_2 is defined by (25), and so (60) follows directly from (26), that is, from the construction of N_0. So we may apply Theorem 1, and we find $u = \Delta x_n$ with $\mathcal{N}_n(\Delta x_n) \leq R_n$ solving (42). Since $\mathcal{N}_n(\Delta x_n) \leq R_n$, we have $\|\Delta x_n\|_\sigma \leq N_n^\sigma R_n = c N_n^{\kappa\beta} \|\bar{y}\|_\delta'$ and $\|\Delta x_n\|_0 \leq R_n = c N_n^{\kappa\beta-\sigma} \|\bar{y}\|_\delta'$, so inequalities (16) and (17) are satisfied. The induction is proved.

The case $\frac{1+\sqrt{5}}{2} \leq \kappa < 2$. The induction hypothesis becomes $\Pi_{N_n}' F(x_n) = \Pi_{N_n}' \bar{y}$. The system (42) (43), (44), 45) becomes:

$$\begin{aligned} f_n(\Delta x_n) &= e_n + \Delta y_n \\ f_n(u) &:= \Pi_{N_{n+1}} \left(F(x_n+u) - F(x_n) \right) \\ e_n &:= \Pi_{N_{n+1}} (\Pi_{N_n} - 1) F(x_n) \\ \Delta y_n &:= \Pi_{N_n} (1 - \Pi_{N_n}) \bar{y} \end{aligned}$$

Using Theorem 1 in the same way, we find that we can find Δx_n satisfying these equations and the estimates (16) and (17) provided:

$$2\,B^{(3)}\,c\,N_n^{\beta}+C_1\left(N_n^{\sigma-\delta}+g^{(\sigma-\delta)_+}N_n^{\kappa(\sigma-\delta)_+-(\delta-\sigma)_+}\right)\leq\gamma c\left(B^{(2)}+2\right)^{-1}g^{-\mu}N_n^{\kappa(\beta-\mu)} \tag{61}$$

But the left-hand side is just $\varphi_1\left(n,N_0\right)N_n^{\kappa(\beta-\mu)}$, where φ_1 is defined by (27), and so (61) follows directly from (28), that is, from the construction of N_0. The induction is proved in this case as well.

4.3 Proof of Proposition 9

We will take advantage of the special form of $e_n:=\Pi_{N_{n+1}}(\Pi_{N_n}-1)F(x_n)$.The proof is the same, with the estimates (55) and (56) derived as follows. Since $x_n\in E_{N_n}$, and E_{N_n} is A-invariant, we have $\Pi_{N_n}Ax_n=Ax_n$ and:

$$\begin{aligned}
e_n &:= \Pi_{N_{n+1}}(\Pi_{N_n}-1)F(x_n)\\
&= \Pi_{N_{n+1}}(\Pi_{N_n}-1)\left(Ax_n+G(x_n)\right)\\
&= \Pi_{N_{n+1}}(\Pi_{N_n}-1)G(x_n)
\end{aligned}$$

Lemma 11 holds for G (though no longer for F), so that estimates (55) and (56) follow readily.

5 An isometric imbedding

We will use the same example as Moser in his seminal paper [14], who himself follows Nash [15]. Suppose we are given a Riemannian structure g^0 on the two-dimensional torus $\mathbb{T}_2=\left(\mathbb{R}/\mathbb{Z}\right)^2$, and an isometric imbedding into Euclidian $\mathbb{R}^5$. In other words, we know $x^0=\left(x_1^0,...,x_5^0\right)$ with:

$$\left(\frac{\partial x^0}{\partial\theta_i},\frac{\partial x^0}{\partial\theta_j}\right)=g_{i,j}^0\left(\theta_1,\theta_2\right)$$

where $g_{i,j}^0=g_{j,i}^0$, so there are three equations for five unknown functions. If we slightly perturb the Riemannian structure, does the imbedding still exist ? If we replace g^0 on the right-hand side by some g sufficiently close to g^0, can we still find some $x:\mathbb{T}_2\to\mathbb{R}^5$ which solves the system ?

We consider the Sobolev spaces $H^s\left(\mathbb{T}_2;\mathbb{R}^5\right)$ and we assume that $x^0\in H^\mu$, with $\mu>3$, and $g^0\in H^\sigma$. Define $F=\left(F^{i,j}\right)$ by:

$$F_{i,j}\left(x+x^0\right)=\left(\frac{\partial}{\partial\theta_i}\left(x+x^0\right),\frac{\partial x}{\partial\theta_j}\left(x+x^0\right)\right)-g^0 \tag{62}$$

Clearly $F\left(0\right)=0$. For $s\geq 3/2$, we know H^s is an algebra, so if $s\geq\mu$ and $x\in H^s$, the first term on the right is in H^{s-1}. On the other hand, the right-hand side cannot be more regular than g^0, which is in H^σ. So F sends H^s

into H^{s-1} for $\mu \leq s \leq \sigma + 1$. The function F is quadratic, hence smooth, and we have:

$$[DF(x)\,u]_{i,j} = \left(\frac{\partial x}{\partial \theta_i}, \frac{\partial u}{\partial \theta_j}\right) + \left(\frac{\partial u}{\partial \theta_i}, \frac{\partial x}{\partial \theta_j}\right) \tag{63}$$

We need to invert the derivative $DF(x)$, that is, to solve the system

$$DF(x)\,u = v_{i,j} \tag{64}$$

It is an undetermined system, since there are three equations for five unknowns. Following Nash, and Moser, we impose two additional conditions:

$$\left(\frac{\partial x}{\partial \theta_1}, u\right) = \left(\frac{\partial x}{\partial \theta_2}, u\right) = 0 \tag{65}$$

Differentiating, and substituting into (63), we find:

$$-2\left(\frac{\partial^2 x}{\partial \theta_i \partial \theta_j}, u\right) = v_{i,j} \tag{66}$$

So any solution φ of (65), (66) is also a solution of (64). The five equations (65), (66) can be written as:

$$M(x(\theta))\,u(\theta) = \begin{pmatrix} 0 \\ v(\theta) \end{pmatrix}$$

with $u = \left(u^i\right), 1 \leq i \leq 5,\ \ v = \left(v^{11}, v^{12}, v^{22}\right)$ and $M(x(\theta))$ a 5×5 matrix. These are no longer partial differential equations. If

$$\det M(x(\theta)) = \det\left(\frac{\partial x}{\partial \theta_1}, \frac{\partial x}{\partial \theta_2}, \frac{\partial^2 x}{\partial \theta_1^2}, \frac{\partial^2 x}{\partial \theta_1 \partial \theta_2}, \frac{\partial^2 x}{\partial \theta_2^2}\right) \neq 0 \tag{67}$$

they can be solved pointwise. Set $L(x(\theta)) := M(x(\theta))^{-1}$, and denote by $M(x)$ and $L(x)$ the operators $u(\theta) \to M(x(\theta))\,u(\theta)$ and $v(\theta) \to L(x(\theta))\,v(\theta)$.

Since $x^0 \in H^\mu$, with $\mu > 3$, we have $x^0 \in C^2$, so $M\left(x^0(\theta)\right)$ is well-defined and continuous with respect to θ. If $\det M\left(x^0(\theta)\right) \neq 0$ for all $\theta \in \mathbb{T}_2$, then there will be some $R > 0$ and some $\varepsilon > 0$ such that $|\det M(x(\theta))| \geq \varepsilon$ for all x with $\left\|x - x^0\right\|_\mu \leq R$. So the operator $L(x)$ is a right-inverse of $DF(x)$ on $\left\|x - x^0\right\|_\mu \leq R$, and we have the uniform estimates, valid on $x^0 + B_\mu(R)$ and $s \geq 0$

$$\begin{aligned}
\|DF(x)\,u\|_0 &\leq C_0 \|x\|_1 \|u\|_1 \\
\|DF(x)\,u\|_s &\leq C_s \left(\|x\|_{s+1} \|u\|_1 + \|x\|_1 \|u\|_{s+1}\right) \\
\|L(x)\,v\|_0 &\leq C_0 \|x\|_\mu \|v\|_0 \\
\|L(x)\,v\|_s &\leq C'_s \left(\|x\|_{\mu+s} \|v\|_0 + \|x\|_\mu \|v\|_s\right)
\end{aligned}$$

This means that the tame estimate (4) is satisfied. However, (5) requires a proof. For this, we have to build a sequence of projectors Π_N satisfying the estimates (1) and (2). For this, we use a multiresolution analysis of $L^2(\mathbb{R})$ (see [13]). Recall that it is an increasing sequence $V_N, N \in \mathbb{Z}$, of closed subspaces of $L^2(\mathbb{R})$ with the following properties:

- $\cap_{N=-\infty}^{N=+\infty} V_N = \{0\}$ and $\cup_{N=-\infty}^{N=+\infty} V_N$ is dense in L^2
- $u(t) \in V_N \Longleftrightarrow u(2t) \in V_{N+1}$
- $\forall k \in \mathbb{Z}, \quad u(t) \in V_0 \Longleftrightarrow u(t-k) \in V_0$
- there is a function $\varphi(t) \in V_0$ such that the $\varphi(t-k), k \in \mathbb{Z}$, constitute a Riesz basis of L^2.

It is known ([13], Theorem III.8.3) that for every $r \geq 0$ there is a multiresolution analysis of $L^2(\mathbb{R})$ such that:

- the $\varphi(t-k), k \in \mathbb{Z}$, constitute an orthogonal basis of V_0
- $\varphi(t)$ is C^r and has compact support: there is some a (depending on r) such that $|t| \geq a \Longrightarrow \varphi(t) = 0$.

We choose r so large that $C^r \subset H^S$. Set $\varphi_N(t) := 2^{N/2}\varphi\left(2^N t\right)$. For N large enough, say $N \geq N_0$, the function φ_N has its support in $]-1/2,\ 1/2[$ so we can consider it as a function on $\mathbb{R}/\mathbb{Z}$, and the $\varphi_{N,k}(\theta) := 2^{N/2}\varphi\left(2^N\theta - k\right)$, for $0 \leq k \leq 2^N - 1$, constitute an orthonormal basis of V_N. In this way, we get a multiresolution analysis on $L^2(\mathbb{R}/\mathbb{Z})$. Setting

$$\Phi_{N,k}(\theta) = 2^N \varphi\left(2^N\theta_1 - k_1\right)\varphi\left(2^N\theta_2 - k_2\right)$$
$$E_N = \operatorname{Span}\left\{\Phi_{N,k}(\theta) \mid k = (k_1, k_2),\ 0 \leq k_1, k_2 \leq 2^N - 1\right\}$$

we get a multiresolution analysis of $L^2(\mathbb{T}_2)$, and the $\Phi_{N,k}(\theta)$ are an orthonormal basis of E_N. Finally, the E_N^5 constitute a multiresolution analysis of $L^2(\mathbb{T}_2)^5 = L^2\left(\mathbb{T}_2, \mathbb{R}^5\right)$. Denote by Π_N the orthogonal projection:

$$(\Pi_N u)^i = \sum_{i,k} < \Phi_{N,k}, u^i > \Phi_{N,k}$$

Introduce an orthonormal basis of wavelets associated with this multiresolution analysis $\left(E_N^5\right)_{N\geq 0}$ of $L^2\left(\mathbb{T}_2, \mathbb{R}^5\right)$. More precisely, the $\Phi_{N_0,k}, 0 \leq k_1, k_2 \leq 2^N - 1$, span E_{N_0}, and one can find a C^r function Ψ with compact support such that the $\Psi_{N,k} = 2^N\Psi\left(2^N\theta_1 - k_1, 2^N\theta_2 - k_2\right)$ span the orthogonal complement of E_{N-1} in E_N. The $\Phi_{N_0,k}$ and the $\Psi_{N,k}$ for $N \geq N_0$ constitute an orthonormal basis for $L^2(\mathbb{T}_2)$. We have, for $u \in L^2(\mathbb{T}_2)$

$$u^i = \sum_k \langle u^i,\ \Phi_{N_0,k}\rangle \Phi_{N_0,k} + \sum_{N\geq N_0,} \sum_k \langle u^i,\ \Psi_{N,k}\rangle \Psi_{N,k}$$

$$\|u\|_{L^2}^2 = \sum_{i=1}^5 \left(\sum_k \langle u^i,\ \Phi_{N_0,k}\rangle^2 + \sum_{n\geq N_0,} \sum_k \langle u^i,\ \Psi_{n,k}\rangle^2\right)$$

It follows from the definition that, for $N \geq N_0$, we have:

$$\|\Pi_N u\|^2 = \sum_{i=1}^{5}\left(\sum_k \langle u^i,\ \Phi_{N_0,k}\rangle^2 + \sum_{N_0 \leq n \leq N,}\ \sum_k \langle u^i,\ \Psi_{n,k}\rangle^2\right)$$

$$\|\Pi_N u - u\|_{L^2}^2 = \sum_{i=1}^{5}\sum_{n \geq N,}\ \sum_k \langle u^i,\ \Psi_{n,k}\rangle^2$$

It is known (see [13] Theorem III.10.4) that:

$$C_1\|u\|_{H^s}^2 \leq \sum_{k \in (K_N)^5} 2^{2N_0 s}\langle u,\ \Phi_{N_0,k}\rangle^2 + \sum_{n \geq N_0,}\ \sum_{k \in (K_n)^5} 2^{2ns}\langle u,\ \Psi_{n,k}\rangle^2 \leq C_2\|u\|_{H^s}^2$$

If $v = \Pi_N u$, we must have $\langle v,\ \Psi_{n,k}\rangle = 0$ for all $n \geq N+1$, so that:

$$C_1\|\Pi_N u\|_{H^s}^2 \leq 2^{2Ns}\|\Pi_N u\|_{L^2}^2 \leq 2^{2Ns}\|u\|_{L^2}^2$$

$$\begin{aligned}\|\Pi_N u - u\|_{L^2}^2 &\leq 2^{-2Ns}\sum_{n \geq N,}\ \sum_{k \in (K_n)^5} 2^{2ns}\langle u,\ \Psi_{n,k}\rangle^2 \\ &\leq 2^{-2Ns}C_2\|u\|_{H^s}^2\end{aligned}$$

So estimates (1) and (2) have been proved.

Finally, we prove (5). We have:

$$\begin{aligned}(\Pi_N u)^i(\theta) &= \sum_k u_k^i \varphi_{N,k}(\theta),\ 1 \leq i < 5 \\ (M(x(\theta))\Pi_N u)^j(\theta) &= \sum_k \sum_i \varphi_{N,k}(\theta)\left(M_i^j(x(\theta))\,u_k^i\right)\end{aligned}$$

So $\Pi_N M(x)\Pi_N$ is a $\left(2^N - 1\right) \times \left(2^N - 1\right)$ matrix of 5×5 matrices $m_{k,k'}$, with:

$$m_{k,k'} = \int_{\mathbb{T}_2} \varphi_{N,k}(\theta)\,\varphi_{N,k'}(\theta)\,M(x(\theta))$$

Looking at the supports of $\varphi_{N,k}$ and $\varphi_{N,k'}$, we see that there is a band along the diagonal outsides which $m_{kk'}$ vanishes.

$$m_{k,k'} = 0 \text{ if } \max_{i=1,2}|k_i - k_i'| > 2^{1-N}a \tag{68}$$

Choose some $\varepsilon > 0$ and N so large that $\max_{i=1,2}\left|\theta - 2^{-N}k_i\right| \leq 2^{1-N}a$ implies that $\left\|M(x(\theta)) - M\left(x\left(2^{-N}k\right)\right)\right\| \leq \varepsilon$. Then:

$$\left\|m_{k,k'} - \int_{\mathbb{T}_2}\varphi_{N,k}(\theta)\,\varphi_{N,k'}(\theta)\,M\left(x\left(2^{-N}k\right)\right)d\theta\right\| \leq \varepsilon\int_{\mathbb{T}_2}|\varphi_{N,k}(\theta)|\,|\varphi_{N,k'}(\theta)|\,d\theta$$

Using the fact that the system $\varphi_{N,k}$ is orthonormal, we get:

$$\left\|m_{k,k'} - \delta_{k,k'} M\left(x\left(2^{-N}k\right)\right)\right\| \leq \varepsilon\left(\max\varphi\right)^2$$

In addition, for every k, (68) gives us at most $4a^2$ non-zero values for k'. So the matrix $m_{k,k'}$ is a small perturbation of the diagonal matrix $\Delta_N := \delta_{k,k'} M\left(2^{-N}k\right),\ k \in K_N$, which is invertible by (67). More precisely, we have:

$$\left(\Pi_N M\left(x\right)\Pi_N\right)^{-1} = \left[I + \Delta_N^{-1}\left(\Pi_N M\left(x\right)\Pi_N - \Delta_N\right)\right]^{-1}\Delta_N^{-1}$$

with $\left\|\Pi_N M\left(x\right)\Pi_N - \Delta_N\right\| \leq \varepsilon 4a^2\left(\max\varphi\right)^2$ and $\left\|\Delta_N^{-1}\right\| \leq \max_\theta\left\|M\left(x\left(\theta\right)\right)^{-1}\right\|$. So $\left(\Pi_N M\left(x\right)\Pi_N\right)$ is invertible for ε small enough, for instance:

$$\varepsilon 4a^2\left(\max\varphi\right)^2\max_\theta\left\|M\left(x\left(\theta\right)\right)^{-1}\right\| < \frac{1}{2}$$

and we have:

$$\left\|\left(\Pi_N M\left(x\right)\Pi_N\right)^{-1}\right\| \leq 2\left\|\Delta_N^{-1}\right\| = 2\left\|\min_\theta M\left(x\left(\theta\right)\right)^{-1}\right\|$$

Estimate (5) then follows immediately

We now introduce the new spaces $X^s := H^{s+\mu}$ and $Y^s := H^{s+\mu-1}$. We denote their norms by $\left\|x\right\|_s^* := \left\|x\right\|_{s+\mu}$ and $\left\|y\right\|_s^* := \left\|y\right\|_{s+\mu-1}$ respectively, so the above estimates become:

$$\begin{aligned}
\left\|DF\left(x\right)u\right\|_s^* &\leq C_s\left(\left\|x\right\|_s^*\left\|u\right\|_0^* + \left\|x\right\|_0^*\left\|u\right\|_s^*\right) \quad \text{for } s \geq 0 \\
u\left\|L\left(x\right)v\right\|_s^* &\leq C_s'\left(\left\|x\right\|_{s+\mu}^*\left\|v\right\|_1^* + \left\|x\right\|_0^*\left\|v\right\|_{s+1}^*\right) \quad \text{for } s \geq 0
\end{aligned}$$

and the range for s becomes $0 \leq s \leq S$ with $S = \sigma + \mu - 1$.We have proved that all the conditions of Corollary 6 are satisfied, with $x^0 \in H^\mu = X^0$, $g^0 \in H^\sigma = Y^{\sigma-\mu+1}$ and $S = \sigma - \mu + 1$. Hence:

Theorem 12 *Take any $\mu > 3$. Suppose $x^0 \in H^\mu$ and $g^0 \in H^\sigma$ with $\sigma \geq 5\mu - 1 > 14$. Suppose the determinant (67) does not vanish. Set $S := \sigma - \mu + 1$. Then, for any δ and α such that:*

$$\begin{aligned}
&\frac{\delta}{\mu} \geq \varphi\left(\frac{S}{\mu}\right) \\
&\frac{\alpha}{\mu} < \min\left\{\frac{\delta}{\mu} - \varphi\left(\frac{S}{\mu}\right),\ \frac{S}{\mu} - \varphi^{-1}\left(\frac{\delta}{\mu}\right)\right\}
\end{aligned}$$

there is some $\rho > 0$ and some $C > 0$ such that, for any g with $\left\|g - g^0\right\|_{\delta+\mu-1} \leq \rho$, there is $x \in H^{\mu+\alpha}$ such that:

$$\begin{aligned}
\left(\frac{\partial x}{\partial\theta_i}, \frac{\partial x}{\partial\theta_j}\right) &= g_{i,j}\left(\theta_1, \theta_2\right) \\
\left\|x - x^0\right\|_\mu &\leq 1 \\
\left\|x - x^0\right\|_{\mu+\alpha} &\leq C\left\|g - g^0\right\|_{\mu+\delta-1}
\end{aligned}$$

Moser [14] finds that if $g, g^0 \in C^{r+40}$ and $x^0 \in C^r$ for some $r \geq 2$, and if $\left|g - g^0\right|_r$ is sufficiently small, then we can solve the problem. Although he made no effort to get optimal differentiability assumptions, we note that our loss of regularity is substantially smaller ($\sigma - \mu \geq 4\mu - 1 > 11$ instead of 40).

Note that when $\mu \to 3$, we have $\sigma \to 14$ and $\delta \to \infty$. In another direction:

Corollary 13 *Suppose $x^0 \in C^\infty$, $g^0 \in C^\infty$, and the determinant (67) does not vanish. Then, for any $\delta > \mu > 3$ and any $\alpha < \delta - \mu$, there is some $\rho > 0$ and some $C > 0$ such that, for any g with $\left\|g - g^0\right\|_{\delta+\mu-1} \leq \rho$, there is some $x \in H^{\alpha+\mu}$ such that:*

$$\left(\frac{\partial x}{\partial \theta_i}, \frac{\partial x}{\partial \theta_j}\right) = g_{i,j}\left(\theta_1, \theta_2\right)$$
$$\left\|x - x^0\right\|_\mu \leq 1$$
$$\|x - x^0\|_{\mu+\alpha} \leq C \left\|g - g^0\right\|_{\mu+\delta-1}$$

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